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Branching rules of semi-simple Lie algebras using affine extensions

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Abstract
We present a closed formula for the branching coefficients of an embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$ of two finite-dimensional semi-simple Lie algebras. The formula is based on the untwisted affine extension of $\mathfrak{p}$. It leads to an alternative proof of a simple algorithm for the computation of branching rules, which is an analogue of the Racah–Speiser algorithm for tensor products. We present some simple applications and describe how integral representations for branching coefficients can be obtained. In the last part, we comment on the relation of our approach to the theory of NIM-reps of the fusion ring in WZW models with chiral algebra $\hat{\mathfrak{g}}_k$. In fact, it turns out that for these models each embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$ induces a NIM-rep at level $k \to \infty$. In cases where these NIM-reps can be extended to finite level, we obtain a Verlinde-like formula for branching coefficients. Reviewing this question, we propose a solution to a puzzle which remained open in related work by Alekseev, Fredenhagen, Quella and Schomerus.

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1. Introduction

Given a module of a Lie algebra $\mathfrak{g}$, it is an important and natural question to ask how this module decomposes under restriction of the action to a subalgebra $\mathfrak{p}$. This decomposition is described by non-negative integer numbers, the so-called branching coefficients. The aim of this paper is to provide new tools for determining branching coefficients in the case where both $\mathfrak{p}$ and $\mathfrak{g}$ are finite-dimensional semi-simple Lie algebras. Several techniques have been developed to deal with this question. Among them are the use of generating functions, Schur functions and a generalization of Kostant's multiplicity formula as well as different kinds of algorithms. For details we refer the reader to [1–5] and references therein.

In this paper, we develop a new approach which uses the fact that a semi-simple Lie algebra $\mathfrak{g}$ is naturally embedded in its affine extension $\hat{\mathfrak{g}}$. This makes available the powerful techniques of affine Kac–Moody algebras (see e.g. [6]) and conformal field theories related to such algebras (see [4] for instance). To give an example, we remind the reader that Verlinde's formula [7] for fusion coefficients in $\hat{\mathfrak{g}}_k$ Wess–Zumino–Witten (WZW) theories gives a generalization of the concept of tensor product coefficients of $\mathfrak{g}$. We will show that analogous relations hold for branching coefficients if we extend either $\mathfrak{g}$ or its subalgebra $\mathfrak{p}$ to the corresponding affine Kac–Moody algebra. In particular, in the first case there exists a relation to the theory of conformal boundary conditions and to the theory of fusion rings in WZW models [8].

The paper is organized as follows. In section 2 we first provide some background on semi-simple Lie algebras and their affine extensions. Subsequently, we present a closed formula for branching coefficients based on the extension of the subalgebra $\mathfrak{p}$ to $\hat{\mathfrak{p}}_k$. This formula is used in turn to give a simple derivation of a Racah–Speiser like algorithm in section 3. Our results are applied to derive properties of branching coefficients and specialized to tensor product coefficients in section 4. In addition, we present a general procedure to obtain integral representations for branching coefficients. As an illustration of this method, we derive an integral representation for branching coefficients of the diagonal embedding $A_1 \hookrightarrow A_1 \oplus A_1$. In section 5 we consider a different approach based on representations of the fusion ring in $\hat{\mathfrak{g}}_k$ WZW models. This leads to a Verlinde-like formula for branching coefficients and induces a second type of integral representations. We exploit the latter to obtain an explicit non-trivial integral representation for the branching coefficients of $A_1 \hookrightarrow A_2$ with embedding index 1. In addition, we indicate that for the A_{2n} series the fusion ring representation contains information about two different embeddings at the same time. This solves a puzzle which remained open in [8].

2. A closed formula for branching coefficients

We want to describe an embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$ of one finite-dimensional semi-simple Lie algebra into another. For notational simplicity, let us assume that $\mathfrak{p}$ actually is a simple Lie algebra but this does not restrict the validity of our results. Denote the weight lattices of $\mathfrak{p}$ and $\mathfrak{g}$ by $\bar{L}_w$ and L_w, respectively. Here and in what follows, we will always use the convention that $i, j, \ldots \in L_w$ and $a, b, \ldots \in \bar{L}_w$. The finite-dimensional irreducible representations of the Lie algebras $\mathfrak{p}$ and $\mathfrak{g}$ are in one-to-one correspondence with the weights with non-negative integral Dynkin labels. These sets of so-called integrable highest weights of $\mathfrak{p}$ are denoted by $\bar{P}^+ \cong \bar{L}_w / W_\mathfrak{p}$ with Weyl group $W_\mathfrak{p}$ and similarly for $\mathfrak{g}$. Let M_a and M_i be the weight systems of the representations $a \in \bar{P}^+$ and $i \in P^+$ including the multiplicities. The embedding can be characterized by a projection $\mathcal{P} : \langle L_w \rangle \to \langle \bar{L}_w \rangle$ where $\langle L \rangle$ means the span of the lattice L over $\mathbb{C}$. Under this projection, the weight system M_i of the representation $i \in P^+$ of $\mathfrak{g}$ decomposes into weight systems of representations of $\mathfrak{p}$ according to

$$\mathcal{P} M_i = \bigoplus_{a \in \bar{P}^+} b_i{}^a M_a. \tag{1}$$

The numbers $b_i{}^a \in \mathbb{N}_0$ are called branching coefficients. Our aim is to find an explicit and general formula for the coefficients $b_i{}^a$ with $i \in P^+$ and $a \in \bar{P}^+$. To achieve this, we consider the untwisted affine extension $\hat{\mathfrak{p}}_k$ of $\mathfrak{p}$. The level k has to be chosen large enough and depends on the value of i. This statement will be made precise below. The integrable highest weights of $\hat{\mathfrak{p}}_k$ are given by the set $\bar{P}^+_k = \bar{L}_w / (W_\mathfrak{p} \ltimes k\bar{L}^\vee)$ where we used the decomposition of the affine Weyl group into a semi-direct product of finite Weyl group and translations by k times the co-root lattice $\bar{L}^\vee$. If we introduce the notation $k(c) = (\theta, c)_\mathfrak{p}$ where θ is the highest root of $\mathfrak{p}$ we may write $\bar{P}^+_k = \{a \in \bar{P}^+ \mid k(a) \leqslant k\}$. The bracket $(\cdot,\cdot)_\mathfrak{p}$ denotes the scalar product on the weight space $\langle \bar{L}_w \rangle$ which is induced by the Killing form. It is given in terms of the quadratic form matrix $F_\mathfrak{p}$ if the weights are written using Dynkin labels, i.e. $(\lambda, \mu)_\mathfrak{p} = \lambda^T F_\mathfrak{p} \mu$. In the following, we will always identify in a natural way an integrable highest weight representation $\hat{c} \in \bar{P}^+_k$ of $\hat{\mathfrak{p}}_k$ with an highest weight $c \in \bar{P}^+$ of $\mathfrak{p} \hookrightarrow \hat{\mathfrak{p}}_k$.

Before we continue, let us briefly introduce further objects that will be needed as we proceed. The character of a highest weight representation $i \in P^+$ of $\mathfrak{g}$ is defined as

$$\chi_i(\cdot) = \sum_{j \in M_i} \mathrm{e}^{(j,\cdot)_\mathfrak{g}} \tag{2}$$

and analogously for $\mathfrak{p}$. The second ingredient of our formula is the modular S matrix of $\hat{\mathfrak{p}}_k$ which, for $a, b \in \bar{P}^+_k$, is given by the Kac–Peterson formula [6]

$$S_{ab} = i^{|\Delta_+|} |\bar{L}_w / \bar{L}^\vee|^{-1/2} (k + g^\vee)^{-r/2} \sum_{w \in W} \epsilon(w) \exp\left\{ -\frac{2\pi \mathrm{i}}{k + g^\vee} (w(a+\rho), b+\rho)_\mathfrak{p} \right\}. \tag{3}$$

This formula involves the rank of the Lie algebra r, the number of positive roots $|\Delta_+|$, the Weyl vector ρ, the dual Coxeter number $g^\vee = (\theta, \rho)_\mathfrak{p} + 1$ and a sum over the Weyl group W including its sign function ϵ. We omit the index $\mathfrak{p}$ because we will not encounter the corresponding objects for the Lie algebra $\mathfrak{g}$. Due to Weyl's character formula, we may write

$$\chi_a(\xi_b) = \frac{S_{ba}}{S_{b0}} \qquad \text{where} \quad \xi_b = -\frac{2\pi \mathrm{i}}{k + g^\vee}(b + \rho) \quad \text{and} \quad a, b \in \bar{P}^+_k. \tag{4}$$

We are now prepared to state the first result of this paper.

Theorem 1. *Consider an embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$ of two finite-dimensional semi-simple Lie algebras. Let $\mathcal{P} : \langle L_w \rangle \to \langle \bar{L}_w \rangle$ be the projection matrix characterizing the embedding and $a \in \bar{P}^+, i \in P^+$ be two arbitrary but integrable highest weights. Define a map $\mathcal{P}^* = F_\mathfrak{g}^{-1} \mathcal{P}^T F_\mathfrak{p} : \langle \bar{L}_w \rangle \to \langle L_w \rangle$ and let k be a number such that $k \geqslant \max\{k(c) \mid {b_i}^c \neq 0\}$. Then we have*

$${b_i}^a = \sum_{d \in \bar{P}^+_k} \sum_{j \in M_i} \bar{S}_{da} S_{d0}\, \mathrm{e}^{-\frac{2\pi \mathrm{i}}{k+g^\vee}(\mathcal{P}j, d+\rho)_\mathfrak{p}} = \sum_{d \in \bar{P}^+_k} \bar{S}_{da} S_{d0} \chi_i(\mathcal{P}^* \xi_d). \tag{5}$$

Proof. For notational simplicity, we assume $\mathfrak{p}$ to be simple. Let us first note that $\max\{k(c) \mid {b_i}^c \neq 0\}$ exists as all weight systems involved are finite. We then start by writing down the identity

$$\sum_{c \in \bar{P}^+_k} {b_i}^c \frac{S_{dc}}{S_{d0}} = \sum_{c \in \bar{P}^+} {b_i}^c \chi_c(\xi_d) = \chi_i(\mathcal{P}^* \xi_d). \tag{6}$$

If we multiply both sides of (6) with $\bar{S}_{da} S_{d0}$ and sum over all $d \in \bar{P}^+_k$ we obtain the desired result due to the unitarity $\sum_d \bar{S}_{da} S_{dc} = \delta^a_c$ of the S matrix. Thus we only have to motivate (6). The left equality simply results from (4) and the condition on the level k, but the right equality is more interesting. Let M_i be the weight system of the representation i including all multiplicities. We insert definition (2) of the characters into (6). After this substitution, the sum on the right-hand side of (6) is over M_i and involves scalar products $(j, \cdot)_\mathfrak{g}$. In contrast to this, the sum in the middle is over the projected weights $\mathcal{P} M_i$ and therefore involves scalar products of the form $(\mathcal{P}j, \cdot)_\mathfrak{p}$. The sum in both cases runs essentially over the same set M_i. Therefore the equality in (6) holds if we can identify the scalar products according to

$(\mathcal{P}j,\cdot)_{\mathfrak{p}} = (j,\mathcal{P}^*\cdot)_{\mathfrak{g}}$. Writing this relation in terms of quadratic form matrices, we see that $\mathcal{P}^*$ was constructed exactly in a way that this identity holds. □

Note the following remarkable observation. If we could rewrite $\mathcal{P}^*\xi_d$ as ξ'_j for some integrable highest weight j of $\hat{\mathfrak{g}}_{k'}$ at a certain level k', we could apply equation (4) and equation (5) would reduce to a Verlinde-like formula [7] for branching coefficients. In general, this does not seem to be possible because $F_{\mathfrak{g}}^{-1}$ might cause negative entries in $\mathcal{P}^*$. We will see however in section 5 that in some specific cases we are able to recover a Verlinde-like formula using a different approach.

Let us briefly comment on the changes if $\mathfrak{p}$ is finite-dimensional and semi-simple but not simple. Under these circumstances, we have a decomposition $\mathfrak{p} \cong \oplus_{s=1}^n \mathfrak{p}_s$ of $\mathfrak{p}$ into simple Lie algebras $\mathfrak{p}_s$. In the affine extension, each simple factor obtains its own level: $\hat{\mathfrak{p}}_k \cong \oplus_{s=1}^n (\hat{\mathfrak{p}}_s)_{k_s}$ with $k = (k_1,\ldots,k_n)$. All relevant structures such as the weight lattice, the Weyl group, the quadratic form matrix and the modular S matrix 'factorize' in some sense, i.e. they are given by a direct sum, a product, a block diagonal matrix or factorize in the original sense of the word. Obviously, the proof of theorem 1 still remains valid if one takes these notational difficulties into account. In particular, the condition $k \geqslant \max\{k(c) \mid {b_i}^c \neq 0\}$ actually means $k_s \geqslant \max\{k_s(c) \mid {b_i}^c \neq 0\}$ in this case.

3. An alternative derivation of a Racah–Speiser like algorithm for branching rules

We will now use formula (5) to give an easy derivation of a well-known algorithm [5] for the calculation of branching coefficients, which is the basis of many computer algebra programs[1]. The algorithm exhibits some similarity with the Racah–Speiser algorithm for the calculation of tensor product multiplicities (see also [6, 9–14] for its extension to fusion rules).

Theorem 2. *Consider an embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$ of finite-dimensional semi-simple Lie algebras. Let $i \in P^+$ be a highest weight of $\mathfrak{g}$ and $\mathcal{P} : \langle L_w\rangle \to \langle \bar{L}_w\rangle$ be the projection matrix characterizing the embedding. The decomposition $\mathcal{P}M_i = \oplus_a {b_i}^a M_a$ can be obtained by the following algorithm*[2].

1. *Calculate the weight system of the representation i including the multiplicities. This gives some set $M_i \subset L_w$.*
2. *Project this set to $\bar{L}_w$ and add the Weyl vector of the subalgebra $\mathfrak{p}$. Now we are dealing with the set $Z_i = \mathcal{P}M_i + \rho \subset \bar{L}_w$ including the multiplicities.*
3. *For each weight of Z_i use a Weyl reflection to map it into the fundamental Weyl chamber where all Dynkin labels are non-negative. An algorithm in terms of elementary Weyl reflections can be found in [3] for example.*
4. *Drop all weights lying on the boundary of the fundamental Weyl chamber and subtract the Weyl vector ρ of the subalgebra $\mathfrak{p}$ from the remaining ones.*
5. *Add up all these contributions including the signs of the relevant Weyl reflections and the multiplicities. The coefficient obtained for each weight $a \in \bar{P}^+$ is just the number ${b_i}^a$.*

Proof. Again we assume $\mathfrak{p}$ to be simple without loss of generality. Essentially, the idea is to evaluate equation (5) for $k \to \infty$. We insert definitions (2) and (3) for the characters and the

[1] I am grateful to M van Leeuwen for providing this information.
[2] The algorithm and the proof are based on [14] in which a slightly different algorithm for calculating NIM-reps for twisted boundary conditions in WZW models is proved.

S matrix. Denoting the prefactor by $\mathcal{N} = |\bar{L}_w/\bar{L}^\vee|^{-1}(k+g^\vee)^{-r}$ we obtain

$$b_i{}^a = \mathcal{N} \sum_{d\in\bar{P}_k^+} \sum_{w_1,w_2\in W} \sum_{j\in M_i} \epsilon(w_1)\epsilon(w_2) \exp\left\{-\frac{2\pi\mathrm{i}}{k+g^\vee}\,(\mathcal{P}j + w_1\rho - w_2(a+\rho),\ d+\rho)_{\mathfrak{p}}\right\} \tag{7}$$

where we have already made use of the defining relation $(j, \mathcal{P}^*\xi_d)_{\mathfrak{g}} = (\mathcal{P}j, \xi_d)_{\mathfrak{p}}$ for $\mathcal{P}^*$. The next step consists in evaluating the sum over d. We define a function $f(d)$ by $b_i{}^a = \sum_{d\in\bar{P}_k^+} f(d+\rho)$. The function $f(c)$ as read from equation (7) has two important properties. First, it satisfies $f(wc) = f(c)$ for all $w \in W$. Indeed, the Weyl reflection may be absorbed into a redefinition[3] of w_1, w_2 and j. To derive the second property let us define the set $\bar{P}^{++}_{k+g^\vee} = \bar{P}^+_k + \rho$. It turns out that $\bar{P}^{++}_{k+g^\vee}$ exactly contains the elements of $\bar{P}^+_{k+g^\vee}$ which do not lie at the boundary of the corresponding affine Weyl chamber. This boundary is given by the set of all weights which are invariant under at least one elementary Weyl reflection including the shifted reflection at the k-dependent hyperplane described by $(\theta,\cdot)_{\mathfrak{p}} = k + g^\vee$. One may show that $f(c) = 0$ if c is invariant under an affine fundamental Weyl reflection. To see this, note that the function $g_x(c) = S_{x,c-\rho}$ which enters $f(d)$ satisfies $g_x(\hat{w}c) = \epsilon(\hat{w})g_x(c)$ with respect to any affine Weyl transformation $\hat{w} \in W \ltimes (k+g^\vee)L^\vee$. These considerations lead to the simple relation

$$b_i{}^a = \frac{1}{|W|} \sum_{d\in\bar{P}_k^+} \sum_{w\in W} f(w(d+\rho)) = \frac{1}{|W|} \sum_{c\in\bar{P}^+_{k+g^\vee}} \sum_{w\in W} f(wc) = \frac{1}{|W|} \sum_{c\in L_w/(k+g^\vee)L^\vee} f(c). \tag{8}$$

We are now in a situation where we are able to perform the sum over $c \in L_w/(k+g^\vee)L^\vee$. The sum over the exponentials in equation (7) exactly gives a non-vanishing result if $\mathcal{P}j + w_1\rho - w_2(a+\rho) \in (k+g^\vee)L^\vee$. In this case, it obviously compensates the normalization factor $\mathcal{N}$. In the limit $k \to \infty$ this condition reduces to a Kronecker symbol and we are left with the k-independent expression

$$b_i{}^a = \frac{1}{|W|} \sum_{w_1\in W} \sum_{w_2\in W} \sum_{j\in M_i} \epsilon(w_1)\epsilon(w_2)\delta_{w_2(a+\rho),\mathcal{P}j+w_1\rho}. \tag{9}$$

Next shift w_2 to the other side of the Kronecker symbol $\left(w_2^{-1} = w_2\right)$ and resum $w_1 \mapsto w_2w_1$ as well as $\mathcal{P}j \mapsto w_2w_1\mathcal{P}j$. The expression under the sum then obviously does not depend on w_2 anymore. By summing over w_2, we compensate the factor $1/|W|$. The final result is

$$b_i{}^a = \sum_{j\in M_i} \sum_{w\in W} \epsilon(w)\delta_{a,w(\mathcal{P}j+\rho)-\rho}. \tag{10}$$

For each weight $\mathcal{P}j + \rho$ lying at the boundary of a Weyl chamber there always exists an elementary Weyl reflection which leaves it fixed. These weights may be omitted because they would contribute twice with different sign. Inserting our result into equation (1) proves the theorem. □

4. Applications and an integral formula for branching coefficients

Using theorem 1 and formula (5) one may explicitly check some well-known properties of branching coefficients. Thus one obtains

[3] Note that the weight system which belongs to an arbitrary representation is invariant under Weyl transformations. In particular this holds for the set $\mathcal{P}M_i$.

Corollary 1. *Let* $\mathfrak{h} \hookrightarrow \mathfrak{p} \hookrightarrow \mathfrak{g}$ *be an embedding of finite-dimensional semi-simple Lie algebras and denote the integrable highest weights by* $\alpha, \beta, \ldots$ *and* $a, b, \ldots$ *and* $i, j, \ldots$ *respectively. The branching coefficients have the following properties.*

1. *The trivial representation* $0 \in P^+$ *decomposes according to* $b_0{}^a = \delta_0^a$.
2. *Denoting the conjugate representation by* $(\cdot)^+$, *the relation* $b_{i^+}{}^{a^+} = b_i{}^a$ *holds.*
3. *The branching coefficients of the embedding* $\mathfrak{h} \hookrightarrow \mathfrak{p} \hookrightarrow \mathfrak{g}$ *are related by* $b_i{}^\alpha = \sum_a b_i{}^a b_a{}^\alpha$.
4. *In the decomposition of a tensor product* $V_i \otimes V_j$ *both reductions are equivalent, i.e. the branching coefficients satisfy* $\sum_l N_{ij}{}^l b_l{}^a = \sum_{c,d} b_i{}^c b_j{}^d N_{cd}{}^a$.

Proof. The first relation holds because $\chi_0(\cdot) = 1$. For the second relation, one needs that the charge conjugation matrix satisfies $C = C^T = C^{-1}$ as well as $F \circ C = C \circ F$ and $C_{\mathfrak{p}} \circ \mathcal{P} = \mathcal{P} \circ C_{\mathfrak{g}}$. The third relation is due to the fact that $\mathcal{P}^*(\mathfrak{h} \hookrightarrow \mathfrak{p} \hookrightarrow \mathfrak{g}) = \mathcal{P}^*(\mathfrak{p} \hookrightarrow \mathfrak{g}) \circ \mathcal{P}^*(\mathfrak{h} \hookrightarrow \mathfrak{p})$. The last property can be checked using the Verlinde formula for $N_{cd}{}^a$ (this is valid if we choose k large enough, see corollary 2), the unitarity of the S matrix and the property $\chi_i \chi_j = \sum_l N_{ij}{}^l \chi_l$ of characters. □

The diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ is special in the sense that its branching coefficients correspond to the tensor products in $\mathfrak{g}$. In this case theorem 1 implies

Corollary 2. *Let* $\mathfrak{g}$ *be a finite-dimensional semi-simple Lie algebra and* V_i, V_k *two fixed integrable highest weight modules. There exists some* $k_0 \in \mathbb{N}$ *such that the coefficients in the decomposition* $V_i \otimes V_j = \oplus_l N_{ij}{}^l V_l$ *may be expressed by the Verlinde formula*

$$N_{ij}{}^l = \sum_{m \in P_k^+} \frac{\bar{S}_{ml} S_{mi} S_{mj}}{S_{m0}}$$

for all integers $k > k_0$.

Proof. This is a simple consequence of theorem 1 and the fact that the branching coefficients for the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ with projection $\mathcal{P}(l_1, l_2) = l_1 + l_2$ are given by the tensor product multiplicities of $\mathfrak{g}$. Using the definition one obtains $\mathcal{P}^*(l) = (l, l)$. The character of $\mathfrak{g} \oplus \mathfrak{g}$ in (5) decomposes into a product of two characters of $\mathfrak{g}$ with argument ξ_l. Applying equation (4) gives the desired result. □

The last remarks concern integral formulae for branching coefficients which may be deduced from theorem 1. We will not give a proof that this is always possible but only give the idea and a simple example for illustration. First we observe that the S matrices and the character in (5) both have a dependence $\sim(d + \rho)/(k + g^\vee)$ on the summation index d. In addition, the two S matrices give a total prefactor of the form $(k + g^\vee)^{-r}$ where r is the rank of the subalgebra, i.e. the number of components of d. Therefore, it is likely that in many (if not all) cases we may rewrite the sum as an integral in the limit $k \to \infty$ and in this way recover an integral representation of branching coefficients.

We show how this works in a very simple example and rederive some integral formula for the (of course well-known) tensor product multiplicities of representations of A_1, i.e. the branching rules of the diagonal embedding $A_1 \hookrightarrow A_1 \oplus A_1$. The characters of A_1 read $\chi_a(x) = \sinh \frac{x}{2}(a+1)/\sinh \frac{x}{2}$ and the S matrix is given by $S_{ab} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2}(a+1)(b+1)$. Using the factorization of the $A_1 \oplus A_1$-character, equation (5) implies for all k greater than some k_0

$$
\begin{aligned}
N_{a_1 a_2}{}^a &= b_{(a_1,a_2)}{}^a \\
&= \frac{2}{k+2}\sum_{b=0}^{k} \frac{\sin\frac{\pi}{k+2}(a+1)(b+1)\ \sin\frac{\pi}{k+2}(a_1+1)(b+1)\ \sin\frac{\pi}{k+2}(a_2+1)(b+1)}{\sin\frac{\pi}{k+2}(b+1)} \\
&= 2\int_0^1 \mathrm{d}x \quad \frac{\sin\pi(a+1)x\ \sin\pi(a_1+1)x\ \sin\pi(a_2+1)x}{\sin\pi x}.
\end{aligned}
$$

For the last equality, we consider the sum to be a Riemann sum with an equidistant partition of the interval $[1/(k+2), (k+1)/(k+2)]$ into intervals of length $\Delta x = 1/(k+2)$. Due to continuity, we may extend the interval to $[0, 1]$. As the integral exists, it is given by the previous series in the limit $k \to \infty$. While such integral representations for general branching coefficients seem to be new, similar statements for tensor products can for example be found in [4, p 534].

5. Relation to conformal field theory and a Verlinde-like formula for branching coefficients

Let us mention that there exists an interesting relation of our work to the classification of boundary conditions in a special class of conformal field theories [4], the so-called WZW models with affine symmetry $\hat{\mathfrak{g}}_k$. It can be shown that for every consistent set of conformal boundary conditions there exists a so-called NIM-rep of the corresponding fusion ring [15]. A NIM-rep is given by non-negative integral matrices $\left(n_i^{(k)}\right)_b{}^a$ satisfying $n_i^{(k)} n_j^{(k)} = \sum_l N_{ij}^{(k)l} n_l^{(k)}$ and $n_{i^+}^{(k)} = \left(n_i^{(k)}\right)^T$ where the numbers $N_{ij}^{(k)l}$ are the fusion rules of the model. One can show that every NIM-rep (at least the finite ones) can be diagonalized by a unitary matrix U and one obtains a Verlinde-like formula of the form

$$
\left(n_i^{(k)}\right)_b{}^a = \sum_d \frac{\bar{U}_{ad} U_{bd} S_{i\phi(d)}}{S_{0\phi(d)}} \tag{11}
$$

with some map $\phi\colon \{a, b, c, d, \ldots\} \to P_k^+$. For recent work on NIM-reps and the connection to the classification of conformal boundary conditions see [15, 16]. Explicit formulae for U may be found in [17, 18]. An approach based on graphs is given in [15]. Note that not all NIM-reps have physical significance [16].

We now want to show how our construction is related to the theory of NIM-reps. Let $\mathfrak{p}$ be a subalgebra of $\mathfrak{g}$. Denote the tensor product multiplicities of $\mathfrak{p}$ by $N_{ab}{}^c$ and the branching coefficients by $b_i{}^a$. One can easily show that the matrices $(n_i)_b{}^a = \sum_c b_i{}^c N_{cb}{}^a$ constitute a NIM-rep of the fusion ring of the WZW model associated with $\hat{\mathfrak{g}}_k$ at level $k \to \infty$. In this limit the fusion rules $N_{ij}{}^l = \lim_{k\to\infty} N_{ij}^{(k)l}$ reduce to the tensor product multiplicities of $\mathfrak{g}$. The proof of the NIM-rep properties relies on the fact that the two possibilities of decomposing a module $V_i \otimes V_j$ of $\mathfrak{g}$ into modules of $\mathfrak{p}$ are equivalent (compare corollary 1) and on the associativity of tensor products. It is easy to generalize the considerations of sections 2 and 3 to obtain

$$
(n_i)_b{}^a = \sum_{c\in\bar{P}^+} b_i{}^c N_{cb}{}^a = \sum_{d\in\bar{P}_k^+} \bar{S}_{da} S_{db} \chi_i(\mathcal{P}^*\xi_d) = \sum_{j\in M_i}\sum_{w\in W} \epsilon(w)\delta_{a,w(\mathcal{P}j+b+\rho)-\rho} \tag{12}
$$

for sufficiently large values of the level k. Note that we did not rely on methods of conformal field theory to obtain this result. We just provided a completely algebraic treatment along the lines of the first four sections.

Our next task is to relate the purely algebraic NIM-reps of the last paragraph to the results from conformal field theory. Indeed, one may prove [8] that NIM-reps which come along

Table 1. Embeddings of simple Lie algebras, known to be related to the limit $k \to \infty$ of NIM-reps of WZW models. The relevant subalgebra is specified by a sequence of maximal embeddings. The statement for the embedding $B_n \hookrightarrow A_{2n}$ is not yet established rigorously.

$\mathfrak{g}$	A_2	A_{2n-1}	A_{2n}	A_{2n}	D_4	D_n	E_6
$\mathfrak{p}$	A_1 $(x_e = 1, 4)$	C_n	$C_n \hookrightarrow A_{2n-1}$	(B_n)	$G_2 \hookrightarrow B_3$	B_{n-1}	F_4

with certain kinds of boundary conditions[4] in $\hat{\mathfrak{g}}_k$ WZW theories coincide with the expressions given in (12) in the limit $k \to \infty$. This means that NIM-reps $\left(n_i^{(k)}\right)_b{}^a$ which may be described as in equation (11) for finite values of k, reduce to expression (12) in the limit $k \to \infty$ for certain distinguished subalgebras $\mathfrak{p}$. In particular, this holds true for the special matrix elements $b_i{}^a = (n_i)_0{}^a$. Starting from (11), we thus obtain another representation of branching coefficients for these distinguished embeddings. On one hand this yields another version of a Racah–Speiser like algorithm [14] invented originally for the calculation of NIM-reps, on the other it may be used to derive alternative integral representations for branching coefficients along the lines of section 4 if one takes the explicit expressions for the matrices U (see for example [18]) and the results of [8] into account.

Table 1 contains a list of embeddings for which these considerations are known or conjectured to be applicable. A large part of these identifications is taken from [8]. Note, that the corresponding subalgebra in almost all examples is given by the subalgebra invariant under the Lie algebra automorphism induced by the Dynkin diagram symmetry to which the NIM-rep belongs. It remained obscure, however, why in the case of $\mathfrak{g} = A_{2n}$ the relevant subalgebra is given by C_n (the so-called orbit Lie algebra [19] of A_{2n}) and not by the subalgebra B_n, invariant with respect to the non-trivial diagram automorphism of A_{2n}. Below, we will partly fill this gap and show that one and the same NIM-rep may lead to two *different* subalgebras under two distinct identifications of NIM-rep labels. We will prove this remarkable feature of NIM-reps in the case of A_2 and comment on the case of A_{2n} with $n > 1$ afterwards. It is an open problem whether all NIM-reps of the type $(n_i)_b{}^a = \sum_c b_i{}^c N_{cb}{}^a$ may be extended to finite values of k. This is certainly true for NIM-reps related to the embeddings given in table 1 (with some caveat regarding embeddings of the type $B_n \hookrightarrow A_{2n}$ for $n > 1$) or to diagonal embeddings $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$, but to our knowledge nothing is known for arbitrary embeddings $\mathfrak{p} \hookrightarrow \mathfrak{g}$.

Let us illustrate our considerations with an example. The Lie algebra $\mathfrak{g} = A_2$ has exactly one automorphism ω related to a non-trivial Dynkin diagram symmetry, where it acts as a permutation of nodes. On the level of weights it thus acts as a permutation of Dynkin labels $\omega(a_1, a_2) = (a_2, a_1)$. As is well known, ω induces a conformal boundary condition in the $\left(A_2^{(1)}\right)_k$ WZW model. Following [18] the boundary labels are given by half-integer symmetric weights $\alpha, \beta = (0, 0), (1/2, 1/2), \ldots, (\lfloor k/2\rfloor/2, \lfloor k/2\rfloor/2)$. Here, the symbol $\lfloor x\rfloor$ denotes the largest integer number smaller or equal to x. The relevant NIM-reps

$$\left(n_{(i_1,i_2)}^{(k)}\right)_\beta{}^\alpha = \sum_{\mu=0}^{\lfloor k/2\rfloor} \frac{\bar{S}^\omega_{\mu\alpha} S^\omega_{\mu\beta} S_{(\mu,\mu),(i_1,i_2)}}{S_{(\mu,\mu),(0,0)}} \tag{13}$$

may be calculated [18] using the explicit formula

$$S^\omega_{\mu\alpha} = \frac{2}{\sqrt{k+3}} \sin \frac{2\pi}{k+3}(\mu+1)(2\alpha+1) \tag{14}$$

where we identified the tupel α with one of its (identical) entries. The obvious similarity of this expression with the S matrix of $A_1^{(1)}$ in mind we may ask whether the NIM-rep (13) in the

[4] These so-called twisted boundary conditions are connected to non-trivial symmetries of the Dynkin diagram of $\mathfrak{g}$.

limit $k \to \infty$ reduces to a NIM-rep of the type (12) coming from an embedding $A_1 \hookrightarrow A_2$. To check this assertion, we have to identify the half-integer symmetric NIM-rep label α, β with weights a, b of A_1 via some map $\Psi : \{a, b, \ldots\} \to \{\alpha, \beta, \ldots\}$. Unfortunately, there are two of these embeddings at our disposal and we have to worry which is the correct one. In [8] a map Ψ has been proposed which leads to the embedding with projection $\mathcal{P}(i_1, i_2) = i_1 + i_2$ and embedding index $x_e = 1$. We will show below, however, that there is another map Ψ' yielding the embedding with projection $\mathcal{P}'(i_1, i_2) = 2(i_1 + i_2)$ and embedding index $x'_e = 4$.

We will discuss the first case first and derive an integral representation for branching coefficients of the embedding $A_1 \hookrightarrow A_2$ with embedding index $x_e = 1$. In this case, one has to use the identification map $\Psi(a) = (a/2, a/2)$ [8]. In order to be able to apply equation (13) we further need the special quotient

$$\frac{S_{(\mu,\mu),(i_1,i_2)}}{S_{(\mu,\mu),(0,0)}} = \frac{\sin\frac{2\pi}{k+3}(i_1+1)(\mu+1) + \sin\frac{2\pi}{k+3}(i_2+1)(\mu+1) - \sin\frac{2\pi}{k+3}(i_1+i_2+2)(\mu+1)}{8\sin^3\frac{\pi}{k+3}(\mu+1)\cos\frac{\pi}{k+3}(\mu+1)}$$

of S matrices of A_2^1 which may be computed using the Kac–Peterson formula (3). Following [8] one may write

$$b_{(i_1,i_2)}{}^a = \lim_{k\to\infty}\left(n^{(k)}_{(i_1,i_2)}\right)_{\Psi(0)}{}^{\Psi(a)} = \lim_{k\to\infty}\sum_{\mu=0}^{\lfloor k/2\rfloor}\frac{\bar{S}^{\omega}_{\mu\Psi(a)}S^{\omega}_{\mu\Psi(0)}S_{(\mu,\mu),(i_1,i_2)}}{S_{(\mu,\mu),(0,0)}}.$$

Performing the continuum limit, we arrive at

$$b_{(i_1,i_2)}{}^a = \int_0^{1/2}\mathrm{d}x\,\frac{\sin 2\pi(a+1)x\,(\sin 2\pi(i_1+1)x + \sin 2\pi(i_2+1)x - \sin 2\pi(i_1+i_2+2)x)}{\sin^2\pi x}.$$

We thus obtained a non-trivial integral formula for the branching coefficients of the embedding $A_1 \hookrightarrow A_2$ with embedding index $x_e = 1$.

As stated above, there is another identification of NIM-rep labels with weights of A_1, leading to the embedding $A_1 \hookrightarrow A_2$ with $x'_e = 4$. We will assume k to be even in what follows. For any even weight a of A_1 define $\Psi'(a) = (k/4, k/4) - (a/4, a/4)$. Before we continue, let us mention two obvious differences compared to the previous identification map Ψ. First, the identification map Ψ' involves the level k explicitly. Second, the map is only well defined for a subset of weights of A_1, i.e. the even ones. One may easily check however, that this restriction corresponds exactly to a general selection rule of the branching coefficients of $A_1 \hookrightarrow A_2$ with $x'_e = 4$. We use our new identification map Ψ' to rewrite (14) according to

$$S^{\omega\prime}_{\mu a} = S^{\omega}_{\mu\Psi'(a)} = \frac{2(-1)^\mu}{\sqrt{k+3}}\sin\frac{\pi}{k+3}(\mu+1)(a+1).$$

Apart from a factor $\sqrt{2}(-1)^\mu$ this is just the S matrix $S^{A_1}_{\mu a}$ of $A_1^{(1)}$ at level $k+1$. Using (4) we are now able to write equation (13) as

$$\left(n^{(k)}_{(i_1,i_2)}\right)_b{}^a = \left(n^{(k)}_{(i_1,i_2)}\right)_{\Psi'(b)}{}^{\Psi'(a)} = 2\sum_{\mu=0}^{k/2}\bar{S}^{A_1}_{\mu a}S^{A_1}_{\mu b}\chi^{A_2}_{(i_1,i_2)}\left(-\frac{2\pi\mathrm{i}}{k+3}(\mu+1,\mu+1)\right).$$

Remembering the definitions of $\mathcal{P}'^*$ in theorem 1 and of ξ_μ in equation (4), the argument of the character can be identified to be $\mathcal{P}'^*\xi_\mu$. By setting the index b to zero, theorem 1 implies

$$b'_{(i_1,i_2)}{}^a = \lim_{k\to\infty}\left(n^{(k)}_{(i_1,i_2)}\right)_0{}^a = \lim_{k\to\infty}\sum_{\mu=0}^{k+1}\bar{S}^{A_1}_{\mu a}S^{A_1}_{\mu 0}\chi^{A_2}_{(i_1,i_2)}(\mathcal{P}'^*\xi_\mu).$$

This equality holds because we are allowed to use the prefactor 2 to extend the range of μ from $0, \ldots, k/2$ to $0, \ldots, k+1$. Taking the considerations of the previous paragraph into account, we just proved that the NIM-rep for the twisted boundary conditions in the $A_2^{(1)}$ WZW model contains information on both embeddings $A_1 \hookrightarrow A_2$, with embedding index $x_e = 1$ or $x_e' = 4$ respectively, at the same time. We leave it to the reader to write down the integral representation for branching coefficients of $A_1 \hookrightarrow A_2$ with $x_e' = 4$.

After the detailed discussion of the A_2 case, we now want to comment on the A_{2n} series for $n > 1$. Numerical analysis indicates that a treatment similar to the one just presented leads to embeddings $B_n \hookrightarrow A_{2n}$, in addition to the embeddings $C_n \hookrightarrow A_{2n}$ which are proposed in [8]. Following [18] the $A_{2n}^{(1)}$ NIM-rep labels are given by fractional symmetric weights α of A_{2n}. To be more concrete, the Dynkin labels have to satisfy the relations $2\alpha_i \in \mathbb{N}_0$, $\alpha_i = \alpha_{2n+1-i}$ and $\sum_{i=1}^{n} \alpha_i \leqslant k/4$. As before, we assume the level k to be even. The map from weights of B_n to the NIM-rep labels is then given by

$$\Psi'(a_1, \ldots, a_n) = \tfrac{1}{4}(2a_{n-1}, \ldots, 2a_1, k - 2a_1 - 2a_2 - \cdots - 2a_{n-1} - a_n, \ldots, 2a_{n-1}).$$

Again this map involves k explicitly and is only well defined for weights satisfying the relevant branching selection rule. We may use the projection $\mathcal{P}(i_1, \ldots, i_{2n}) = (i_1 + i_{2n}, i_2 + i_{2n-1}, \ldots, 2(i_n + i_{n+1}))$ to calculate the branching rules of $B_n \hookrightarrow A_{2n}$ according to theorem 2 and compare them to NIM-rep calculations at $k \to \infty$ which have been performed using the algorithm proved in [14]. Taking our new identification of subalgebra weights with NIM-rep labels into account, full agreement has been observed. Up to now, however, we have no rigorous proof to support this observation. As a last remark, note that even in the case of A_4 our new identification requires a maximally embedded $B_2 \cong C_2$ in contrast to the result in [8].

6. Conclusions

In our paper, we derived an explicit formula for the branching rules of embeddings of two semi-simple Lie algebras. Starting from this result, we gave an alternative proof for an algorithm which can be used to calculate branching rules. We have also been able to check some simple properties of branching coefficients explicitly and argued that our formula induces integral representations for them. In two examples, these integral representations have been derived explicitly. Finally, we discussed the relation of embeddings to NIM-reps of WZW models at infinite level. In particular, we solved a puzzle which remained open in [8] and found that one NIM-rep may contain informations about several embeddings at the same time by re-interpretation of NIM-rep labels. A possible continuation of general NIM-reps of type (12) to finite values of k using a Verlinde-like formula (11) might be important for a representation theoretic understanding of embeddings of quantum groups at roots of unity as it provides a natural analogue to the transition from tensor product to fusion coefficients (cf [12, 13]). This last point has to be clarified in future work. Note that there has been some progress recently in understanding subgroups of quantum groups [20–24].

Another approach to express the branching coefficients of semi-simple Lie algebras by using affine extensions of both Lie algebras at the same time would be to consider the grade zero part of the corresponding branching functions. A general expression for branching functions was found in [25]. However, it does not seem to provide a considerable simplification in our context.

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Note added in proof. After submission of this article, two preprints [26, 27] have been published which discuss the relation of NIM-reps in $\mathfrak{g}_k$ WZW models to certain subalgebras of $\mathfrak{g}$ and their affine extensions for finite values of the level k.

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